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Phase Error and Stability of Second Order Methods for Hyperbolic Problems. I

ELI TURKEL

Mathematical Applications Group, Inc.

Elmsford, N.Y. 10523

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Second order schemes for hyperbolic systems are compared with respect to stability properties and minimization of phase errors. Among the schemes that utilize no points beyond a nine point lattice those with the smallest phase errors are a Strang type splitting scheme, Leapfrog, and Lax-Wendroff (written as a two-step scheme). Because of its optimal permissible time step the time splitting scheme is to be preferred. Both the Burstein and MacCormack schemes are found to be weakly unstable and have larger phase errors while the rotated Richtmyer method has a much larger phase error than these other schemes. It is possible, however, to reduce the phase error by using points beyond the eight nearest neighbors. However, the Richtmyer method together with its generalization by Gourlay and Morris still have very large phase errors. The method of Fromm together with the SHASTA code are nonlinear schemes that do have reduced phase errors. However, because of the smaller permissible time steps coupled with additional complexities it would seem more worthwhile to use higher order schemes once one is willing to use data beyond the nine point rectangular mesh. Numerical experiments, with vector equations, are presented that confirm these results.

I. INTRODUCTION

In this paper we shall analyze the stability properties and phase errors of several two-dimensional second-order schemes. In order to limit the scope of the study we shall consider only explicit multistep schemes that depend on data at the previous time level. The use of implicit schemes is generally not worth the additional computational effort for nonlinear hyperbolic problems. We shall concentrate on schemes that use information within a nine point rectangular lattice at the previous time level. Schemes that use mesh points beyond this lattice present additional difficulties near boundaries, especially curved boundaries. Furthermore, for problems where boundaries present no difficulties, e.g. periodic boundary conditions, it is usually more advantageous to use methods of higher order than second order. Nevertheless several schemes that use points beyond this lattice will be mentioned as they have some interesting properties and are occasionally used in

practice. Thus, the schemes that we shall discuss can be viewed as multidimensional generalizations of the Lax-Wendroff method, though some comparisons with the Leapfrog method will be made.

We shall consider the general equation

$$w_t + f_x + g_y = 0, \tag{1}$$

where u, f, g are vectors. Equation (1) can also be written as a quasilinear system of equations

$$w_t + Aw_x + Bw_y = 0, \tag{2}$$

where we shall assume that A and B are simultaneously symmetrizable. We shall define the phase error only for the case where A and B commute, though the results seem to be applicable to more general systems. In fact, the matrices A and B do not commute in the sample problem used for testing these schemes. When A and B do commute we define the numerical phase as

$$P(\xi, \eta) = -\arctan[(\text{Im } G)(\text{Re } G)^{-1}], \tag{3}$$

where ξ, η are the Fourier variables and $G(\xi, \eta)$ is the amplification matrix.

We shall use the amplification matrix G and phase P only for the linearized version of Eq. (2), and we shall not include the effects of boundary conditions (except in the result section). For all finite-difference methods the true phase is approximated reasonably well only for the long-wave components of the solution while the higher waves are poorly represented. Thus, $P(\xi, \eta)$ is of importance only for ξ and η small. We define λ as $\Delta t/\Delta x$ and σ as $\Delta t/\Delta y$. Then, by consistency we have for second order schemes that

$$G(\xi, \eta) = I - i(\lambda A\xi + \sigma B\eta) - \tfrac{1}{2}(\lambda A\xi + \sigma B\eta)^2 + iH(\xi, \eta) + O(\xi^4 + \eta^4), \tag{4}$$

where H is a homogeneous polynomial of degree three. Furthermore, if only centered differences are used then H is a real matrix. We therefore, have

$$\tan P(\xi, \eta) = [-(\lambda A\xi + \sigma B\eta) + H(\xi, \eta)] \cdot [I - \tfrac{1}{2}(\lambda A\xi + \sigma B\eta)^2]^{-1} + O(\xi^4 + \eta^4). \tag{5}$$

When the matrices A and B commute we can expand the inverse appearing in Eq. (5) in a Neumann series. Then Eq. (5) can be written as

$$\tan P(\xi, \eta) = -(\lambda A\xi + \sigma B\eta) + H(\xi, \eta) - \tfrac{1}{2}(\lambda A\xi + \sigma B\eta)^3 + O(\xi^4 + \eta^4). \tag{6}$$

We define $P_A(\xi, \eta)$ as the analytic phase. Thus, we have

$$P_A(\xi, \eta) = -(\lambda A\xi + \sigma B\eta). \tag{7}$$

Combining these expressions we have the following formula for the numerical phase error.

$$\begin{aligned} E(\xi, \eta) &= P(\xi, \eta) - P_A(\xi, \eta) \\ &= (-(\lambda A\xi + \sigma B\eta) - \tfrac{1}{2}(\lambda A\xi + \sigma B\eta)^3 + H(\xi, \eta) + \tfrac{1}{3}(\lambda A\xi + \sigma B\eta)^3) \\ &\quad + (\lambda A\xi + \sigma B\eta) + O(\xi^4 + \eta^4), \end{aligned}$$

where we have also included cubic terms in the expansion of the arctangent function in Eq. (6). This expression reduces to

$$E(\xi, \eta) = H(\xi, \eta) - \tfrac{1}{6}(\lambda A\xi + \sigma B\eta)^3 + O(\xi^4 + \eta^4) \tag{8}$$

We note that when E is positive for ξ, η positive then the numerical solution lags behind the analytic solution.

Before we describe any schemes we introduce some space saving notation. Let μ_x and δ_x be the averaging and difference operators in the x direction defined by

$$\mu_x w_{i,j} = \tfrac{1}{2}(w_{i+1/2,j} + w_{i-1/2,j}), \qquad \delta_x w_{i,j} = w_{i+1/2,j} - w_{i-1/2,j},$$

with a similar notation for the operators in the y direction. It then follows that

$$\delta_x \mu_x w_{i,j} = \tfrac{1}{2}(w_{i+1,j} - w_{i-1,j}).$$

II. Nine-Point Schemes

1. As the first scheme we analyze the original Lax–Wendroff method [17] which can also be written as a two-step method (see Thommen [26], Singleton [24]).

$$\begin{aligned} \bar{w}^{(1)}_{i+1/2,j} &= \mu_x w - \tfrac{1}{2}\lambda\delta_x f - \tfrac{1}{2}\sigma\mu_x\delta_y\mu_y g, \\ \bar{w}^{(2)}_{i,j+1/2} &= \mu_y w - \tfrac{1}{2}\lambda\mu_y\delta_x\mu_x f - \tfrac{1}{2}\sigma\delta_y g, \\ w^{n+1}_{i,j} &= w^n_{i,j} - \lambda\delta_x \bar{f}^{(1)} - \sigma\delta_y \bar{g}^{(2)}. \end{aligned} \tag{9}$$

We then have that

$$\begin{aligned} G(\xi, \eta) &= I - i(\lambda A \sin\xi + \sigma B \sin\eta) \\ &\quad - \left[\lambda^2 A^2(1 - \cos\xi) + \sigma^2 B^2(1 - \cos\eta) + \lambda\sigma\left(\frac{AB + BA}{2}\right)\sin\xi \sin\eta\right]. \end{aligned} \tag{10}$$

This method is stable for $\rho(A, B) \leqslant 1/\sqrt{8}$ (see [17]). We also have

$$E(\xi, \eta) = \tfrac{1}{6}(\lambda A\xi^3 + \sigma B\eta^3 - (\lambda A\xi + \sigma B\eta)^3) + O(\xi^4 + \eta^4). \tag{11}$$

It is evident that if $\lambda\rho(A) + \sigma\rho(B) < 1$ and ξ and η are positive then E is positive, i.e., there is a phase lag. We shall use Eq. (11) as a basis for comparison with the other schemes to be described.

2. MacCormack [19] introduced several schemes that use data at only seven points from the nine point lattice. A sample scheme from this group is

$$\begin{aligned} \bar{w}_{i,j} &= w^n_{i,j} - \lambda\delta_x f^n_{i+\frac{1}{2},j} - \sigma\delta_y g^n_{i,j+\frac{1}{2}}, \\ w^{n+1}_{i,j} &= \frac{1}{2}(w^n_{i,j} + \bar{w}_{i,j}) - \frac{\lambda}{2}(\bar{f}_{i,j} - \bar{f}_{i-1,j}) - \frac{\sigma}{2}(\bar{g}_{i,j} - \bar{g}_{i,j-1}). \end{aligned} \tag{12}$$

The amplification matrix associated with this scheme is

$$\begin{aligned} G(\xi, \eta) = I - i(\lambda A \sin\xi + \sigma B \sin\eta) - \Big[&\lambda^2 A^2(1 - \cos\xi) + \sigma^2 B^2(1 - \cos\eta) \\ &+ 2\lambda\sigma \sin\frac{\xi}{2}\sin\frac{\eta}{2}\left(\cos\left(\frac{\xi - \eta}{2}\right)(AB + BA)\right. \\ &\left. - i\sin\left(\frac{\xi - \eta}{2}\right)(AB - BA)\right)\Big]. \end{aligned} \tag{13}$$

This scheme has the advantage of requiring few operations. However, for many matrices A and B the scheme is unconditionally unstable. For example if $\lambda A = -\sigma B$ then

$$|G(\xi, -\xi)|^2 = I + (\lambda A \sin\xi)^4 > I.$$

Rotation of the four variant schemes described by MacCormack is no guarantee of stability either. MacCormack remarks that if one considers $\Delta t = O((\Delta x)^{4/3})$ then the scheme is stable. However, with this type of assumption most weakly unstable schemes would become stable (see Richtmyer and Morton, p. 231 for a discussion of this apparent paradox). Furthermore, the stability condition given by MacCormack [18] is not dimensionally correct, i.e., the stability condition depends on the units used which is obviously not desirable. Nevertheless in many situations the instability may be weak enough so that the equations can be integrated numerically especially if some viscosity is added. Examples of this behavior will be shown in connection with other schemes. However, should this scheme be used one must always be careful of possible linear instabilities. With this scheme we have a phase error

$$\begin{aligned} E(\xi, \eta) = &\tfrac{1}{6}(\lambda A\xi^3 + \sigma B\eta^3) + \tfrac{1}{4}\lambda\sigma\xi\eta(\xi + \eta)(AB - BA) \\ &- \tfrac{1}{6}(\lambda A\xi + \sigma B\eta) + O(\xi^4 + \eta^4). \end{aligned} \tag{14}$$

For the case where A and B commute this expression is identical with that obtained for the Lax–Wendroff method. However, when A and B do not commute (and the numerical phase is not analytically defined) the computational results, given later, indicate that the MacCormack scheme has a larger phase lag than the Lax–Wendroff method.

3. The next scheme that we consider is actually a two parameter family of schemes. This family is a generalization of a method introduced by Burstein [3]:

$$\begin{aligned} \bar{w} &= \mu_x\mu_y w^n - \alpha[\lambda\mu_y\delta_x f^n + \sigma\mu_x\delta_y g], \\ w^{n+1} &= w^n - \nu\delta_x^2\delta_y^2 w^n - (1/2\alpha)(\lambda\mu_y\delta_x\bar{f} + \sigma\mu_x\delta_y\bar{g}) \\ &\quad - (1 - 1/2\alpha)(\delta_x\mu_x f^n + \delta_y\mu_y g^n). \end{aligned} \tag{15}$$

In this class of schemes we predict, to first order, a temporary value of w at time $t + \alpha\,\Delta t$ and then average to obtain second order accuracy at $t + \Delta t$. Burstein introduced the schemes with $\nu = 0$ and $\alpha = \frac{1}{2}$ (in Eq. (15)) which, following Wilson [27], we shall call the rotated Richtmyer method, and also $\nu = 0$, $\alpha = 1$, which has been used extensively for many different problems. The amplification matrix associated with this family is

$$\begin{aligned} G(\xi, \eta) = I &- 16\nu \sin^2\frac{\xi}{2}\sin^2\frac{\eta}{2} + i\left(\lambda A\left(1 - \frac{1}{4\alpha}(1 - \cos\eta)\right)\sin\xi\right. \\ &\left. + \sigma B\left(1 - \frac{1}{4\alpha}(1 - \cos\xi)\right)\sin\eta\right) - \frac{1}{2}(\lambda^2A^2(1 - \cos\xi)(1 + \cos\eta) \\ &+ \sigma^2B^2(1 - \cos\eta)(1 + \cos\xi) + \lambda\sigma(AB + BA)\sin\xi\sin\eta). \end{aligned} \tag{16}$$

If ν is positive but less than $1/16$ then the family is dissipative and hence stable for some range of the eigenvalues of A and B. However, if ν is zero then this family can be unstable except for the rotated Richtmyer method. In fact

$$G(\pi/2, \pi) = I - (\sigma B)^2 + i(1 - 1/2\alpha)\,\lambda A.$$

Thus, if λA is in some sense large compared with σB we shall have an eigenvalue of G which is greater than 1, unless $\alpha = 1/2$. In particular if $\sigma B = \epsilon\lambda A$ then

$$|\,G(\pi/2, \pi)|^2 = 1 - (2\epsilon^2 - (1 - 1/2\alpha)^2)(\lambda A)^2 + \epsilon^4(\lambda A)^4$$

and the schemes are unstable when $\epsilon \leqslant 1/\sqrt{2}(1 - 1/2\alpha)$. For the rotated Richtmyer case both Wilson and Zwas have shown that it is stable if $\rho(\lambda A, \sigma B) \leqslant 1/\sqrt{2}$ where $\rho(A, B)$ denotes the larger of the spectral radii of A and B [27, 28]. As with the MacCormack method the instabilities are weak. Burstein has used his scheme,

$\alpha = 1$, in some cases for tens of thousands of time cycles without any artificial viscosity (see for example [4]). In other cases an artificial viscosity was required for long term stability. For the particular case of the fluid dynamic matrices we have not been able to estimate the amplification matrix for the complete range of dependent values. However for hypersonic flow $u, v \gg c$ we can show that the scheme is unstable if $(1/\sqrt{8})\,\lambda u < \sigma v$ or $(1/\sqrt{8})\,\sigma v < \lambda u$.

In particular, we tried this scheme on the following test problem.

$$u_t + \tfrac{1}{2}(u_x + u_y) = 0, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1;$$
$$u(x, y, 0) = f(x, y), \qquad u(0, y, t) = u(1, y, t), \qquad u(x, 0, t) = u(x, 1, t).$$

The solution of this equation is $u = f(x - \frac{1}{2}t, y - \frac{1}{2}t)$. A mesh of 80×20, with $\sigma B = 1/4$ was chosen so that with $\alpha = 1$ we have $|G(\pi/2, \pi)|^2 = 1.016$. When the initial condition was chosen as $\sin(x + y)$, no instabilities were noticed after 10 complete periods were traversed (about 1600 cycles). With random initial data there was a linear growth in the energy for about 500 cycles when exponential growth set in. Thus, even with random data, which is worse than that which occurs in physical problems, the instabilities are slight and can be controlled by an artificial viscosity. In fact the mild instability can be beneficial in preventing excessive damping from overwhelming the solution. Nevertheless, one must be careful to avoid linear instabilities. It is impossible to verify whether the instabilities that occur in the blunt body problem are nonlinear instabilities as Burstein claims or are in fact linear instabilities. Furthermore, if the scheme is even mildly unstable convergence is no longer guaranteed even for numerical solutions that display no energy growth (see Richtmyer and Morton [23]). As was to be expected the rotated Richtmyer method showed no instabilities even after 1600 cycles and with random initial data.

Fromm [9] has suggested that since most schemes have phase lags one should predict the solution at a future time and then average. This reasoning indicates that to decrease the phase error one should choose a large value of α. This in fact is the justification for considering values of α that are greater than 1. The phase error for Eq. (15) is

$$E(\xi, \eta) = \lambda A\left(\frac{\xi^3}{6} + \frac{\xi\eta^2}{8\alpha}\right) + \sigma B\left(\frac{\eta^3}{6} + \frac{\xi^2\eta}{8\alpha}\right) - \frac{1}{6}(\lambda A\xi + \sigma B\eta)^3 + O(\xi^4 + \eta^4). \tag{17}$$

We notice that the phase error for this family is greater than that for the Lax–Wendroff method for all α. However, as seen in Fig. 2 we can achieve better phase representations by choosing ν greater than zero and α sufficiently large (even though these involve fourth order terms). Thus, if one is willing to accept some artificial fourth order damping a smaller phase error than that given by

Lax–Wendroff can be achieved. In addition by choosing ν positive we have gained linear stability. Though this term can also be added to the Lax–Wendroff scheme it is not to be recommended since there will be a large amount of damping in the scheme.

Equation (17) shows that the phase error decreases as we increase α. However, at the same time we decrease the allowable step size. There seems to be general correlation in these types of schemes that better stability properties implies a poorer representation of the correct velocities. This can be seen heuristically from Eqs. (4) and (8). Thus, for stability we would want $H(\xi, \eta)$, the third order terms, to be more positive while for improved phase we would want H to be more negative. However, as will be shown there are exceptions to this rule.

4. A popular class of methods which allow maximal time steps are based on the concept of time splitting as introduced by Strang [25]. Eilen, Gottlieb and Zwas [7] have suggested one such efficient scheme which uses only the nine-point lattice. For the basic scheme one can use a one dimensional version of the generalized Burstein method (Eq. (15)) as suggested by McGuire and Morris [20]. Thus, the one dimensional scheme to be considered is

$$\begin{aligned} \bar{w} &= \mu_x w - \alpha\lambda\delta_x f^n \\ w^{n+1} &= w^n - (1/2\alpha)\,\lambda\delta_x \bar{f} - (1 - 1/2\alpha)\,\delta_x\mu_x f^n \end{aligned} \tag{18}$$

and we denote this method in operator notation as

$$w^{n+1} = L_x(w^n).$$

We can then construct a two dimensional method by letting

$$w^{n+1} = L_x L_y w^n.$$

However, this method would be of second order only if the matrices A and B commute. Therefore, Eilon, Gottlieb and Zwas suggest a slight modification of the Strang $L_{x/2}L_yL_{x/2}$ scheme, i.e., alternating the order of the operators at each time step. Thus

$$w^{n+2} = L_x L_y L_y L_x w^n. \tag{19}$$

This method is of second order after an even number of steps for all A and B. In fact since the error for one step is one order higher than the accumulated error this method is of second order for all time steps.

Note, that in contradistinction to the generalized Burstein scheme the amplification matrix associated with Eq. (19) does not depend on the parameter α appearing in Eq. (18). The dependence on α appears only in the nonlinear terms. Since this

effect is problem-dependent and usually of much smaller significance than the linear terms we shall confine our attention to the linearized version of Eq. (19) in which case the parameter α no longer appears. The amplification matrix (for one time step) for this scheme is the product of two matrices each derived from a one dimensional scheme and so this scheme, as with all time splitting schemes, has maximal stability properties, i.e., $\rho(\lambda A, \sigma B) \leqslant 1$. The phase error for this scheme is

$$E(\xi, \eta) = \tfrac{1}{6}(\lambda A\xi^3 + \sigma B\eta^3) - \tfrac{1}{6}(\lambda A\xi + \sigma B\eta)^3 + \tfrac{1}{2}(\lambda A)^2 \sigma B\xi^2\eta + \tfrac{1}{2}(\lambda A)(\sigma B)^2 \xi\eta^2 + O(\xi^4 + \eta^4). \tag{20}$$

As before, we assume that A and B commute. We can simplify this formula to obtain

$$E(\xi, \eta) = \tfrac{1}{6}[(\lambda A - (\lambda A)^3)\,\xi^3 + (\sigma B - (\sigma B)^3)] + O(\xi^4 + \eta^4). \tag{21}$$

By the stability condition we have that $\rho(\lambda A) \leqslant 1$ and so $\rho(\lambda A - (\lambda A)^3) \leqslant 1$. Hence we have a phase lag as we had with the other schemes analyzed thus far. Also, in the scalar case if A equals B we can choose the time step so that $\lambda A = 1$ and the phase error is of fourth order. For small time steps this error is similar to that of the Lax–Wendroff scheme (Eq. (11)) since the third order terms in Δt are small. For larger time steps the Lax Wendroff scheme is no longer stable while the phase error for the time splitting method decreases. Thus, for most problems the time splitting method as given by (18, 19) seems to give comparable phase errors to the Lax–Wendroff method while allowing a larger allowable time step. Both schemes display comparable speed per time step.

5. For purposes of future reference we record the amplification matrix and phase error for the nonstaggered Leapfrog method.

$$w^{n+1} = w^{n-1} - \lambda\delta_x\mu_x f^n - \sigma\delta_y\mu_y g^n, \tag{22}$$

$$G(\xi, \eta) = -i(\lambda A \sin\xi + \sigma B \sin\eta) + \operatorname{sqrt}[1 - (\lambda A \sin\xi + \sigma B \sin\eta)^2],$$

$$E(\xi, \eta) = \tfrac{1}{6}(\lambda A\xi^3 + \sigma B\eta^3) - \tfrac{1}{6}(\lambda A\xi + \sigma B\eta)^3 + O(\xi^4 + \eta^4). \tag{23}$$

We see that within fourth-order terms this phase error is identical with that for the Lax–Wendroff method. In Fig. 1 we see that for the special case of $\lambda A = \sigma B = 1/4$ the phase errors for both these schemes are extremely close for all $\xi = \eta$. In addition, the Leapfrog scheme has the stability condition $\rho(\lambda A + \sigma B) \leqslant 1$ which is better than the Lax–Wendroff condition but not as good as the splitting techniques.

III. Other Schemes

The schemes that we have considered thus far all have a phase error of the form

$$E(\xi, \eta) = \tfrac{1}{6}[(\lambda A)\,\xi^3 + (\sigma B)\,\eta^3 - (\lambda A\xi + \sigma B\eta)^3] + P_1(A, B)\,\xi^2\eta + P_2(A, B)\,\xi\eta^2 + O(\xi^4 + \eta^4), \tag{24}$$

where P_i are polynomials in their arguments. If one wishes to change the leading terms in this expression then one must use data beyond the nine-point lattice, in particular one must use data beyond three mesh points in the x and y directions independently. As previously mentioned an obvious choice is to use third- or fourth-order methods. However, in this study we have confined our attention to second-order schemes.

6. Fromm [9] introduced a new scheme with the specific purpose of reducing the phase error by using data beyond the nine-point lattice. He considers the time splitting scheme L_xL_y but modifies the one dimensional scheme. For a scalar equation the operator L_x given in Eq. (18) is replaced by

$$w^{n+1} = w^n - AT\mu^2\,\delta w + \tfrac{1}{4}(A^2\,\delta^2 w + (A - 2A)\,T^2\,\delta^2 w), \tag{25}$$

where T is a shift operator defined by

$$Tw_j = \begin{cases} w_{j+1/2}\,, & A > 0, \\ w_{j-1/2}\,, & A < 0, \end{cases}$$

and hence this is a nonlinear one dimensional operator which uses data at four mesh points at the previous time level. As before if one wishes to extend this scheme to vector equations using noncommuting matrices one must alternate the order of the operators L_x, L_y at each time step. The one dimensional amplification matrix for this scheme, with $A > 0$, is

$$G(\xi) = I - \tfrac{1}{2}\lambda A[\sin^2\xi + \lambda A(1 - \cos\xi) + (\lambda A - 2)(1 - \cos\xi)] - \tfrac{1}{2}i\lambda A \sin\xi[1 + \cos\xi - (\lambda A - 2)(1 - \cos\xi)]. \tag{26}$$

Fromm does not present any stability analysis for this scheme; however, choosing $\xi = \pi$ we have that

$$G(\pi) = I - \tfrac{1}{2}\lambda A(2\lambda A - 2(\lambda A - 2)) = I - 2\lambda A.$$

Thus a necessary condition for stability is that $\lambda\rho(A) \leqslant 1/2$ and similarly for B. The phase error for the two dimensional scheme is

$$E(\xi, \eta) = -\tfrac{1}{12}[\lambda A(1 - \lambda A)(1 - 2\lambda A)\,\xi^3 + \sigma B(1 - \sigma B)(1 - 2\sigma B)\,\eta^3] + O(\xi^4 + \eta^4). \tag{27}$$

Thus, this scheme has a phase gain. Compared with the unmodified time splitting version (Eq. (21)) we have replaced the factor $(1/6)(1 + \lambda A)$ by $-(1/12)(1 - 2\lambda A)$. Hence there is the benefit of a smaller coefficient together with a matrix term which vanishes identically for $\lambda A = (1/2)I$ (similarly for the matrix B) and should be, in general, smaller for time steps within the stability range. Thus, Fromm has achieved a slightly smaller phase error but at the expense of requiring a larger domain of dependence, a nonlinear scheme and a smaller allowable time step.

7. Another nonlinear method we mention is embodied in the SHASTA code introduced by Boris and Book [2]. The one-dimensional version uses a seven-point lattice at the previous time level; however, it is second order for the fluid dynamic equations only in the case of uniform flow. As with the Fromm scheme there is a smaller phase error but at the expense of a larger domain of dependence, which can cause difficulties near boundaries. Furthermore, at least for the scalar case with a linearized difference scheme, the permissible time step is one half that of the splitting methods. In addition the SHASTA code requires about two and a half times as much computational time per time step as does the optimized scheme [Eqs. (18, (19)]. It would therefore seem that for problems where accurate velocity approximations are important that a higher order scheme would yield better approximations with less computer time and a smaller domain of dependence. Nevertheless, this method has the property of introducing additional compression (rather than the usual diffusion) which can be useful for fluid dynamic problems requiring a sharply defined shock front.

8. In introducing the Leapfrog scheme with a staggered mesh we have another scheme whose one dimensional version extends beyond the three-point lattice and hence the two dimensional version extends beyond the nine point rectangular lattice. For this scheme

$$w^{n+1} = w^n - \tfrac{1}{2}(\mu_x \delta_x f^{n+1/2} + \mu_y \delta_y g^{n+1/2}). \tag{28}$$

The amplification matrix of this scheme is

$$G(\xi, \eta) = I - \tfrac{1}{2}(\lambda A \sin \xi + \sigma B \sin \eta)^2 + i(\lambda A \sin \xi + \sigma B \sin \eta) \operatorname{sqrt}(1 - \tfrac{1}{4}(\lambda A \sin \xi + \sigma B \sin \eta)^2), \tag{29}$$

and the phase error is

$$E(\xi, \eta) = \tfrac{1}{6}(\lambda A \xi^3 + \sigma B \eta^3) - \tfrac{1}{12}(\lambda A \xi + \sigma B \eta)^3 + O(\xi^4 + \eta^4). \tag{30}$$

The phase error for this scheme differs from that of the nonstaggered Leapfrog method in terms that depend on $(\lambda A)^3$, $(\sigma B)^3$. Thus, for small time steps the two sets of phase errors are similar but for larger time steps the staggered Leapfrog

has a slightly smaller phase lag. However, this method is at a disadvantage near boundaries where half the variables are defined at the boundary while another half are defined a mesh width away. However, for problems with periodic boundary conditions where dissipation is not needed (e.g., global circulation models) this is probably the best of the second-order methods (see Morton [22]).

9. Kreiss and Oliger [15] have suggested an extension of the Leapfrog method which is second order in time but fourth order in space. Thus,

$$w^{n+1} = w^{n-1} - \lambda(10\delta_x\mu_x - 4\delta_x\mu_x^3) f^n - \sigma(10\delta_y\mu_y - 4\delta_y\mu_y^3) g^n. \tag{31}$$

Then,

$$E(\xi, \eta) = -\tfrac{1}{6}(\lambda A\xi + \sigma B\eta)^3 + O(\xi^4 + \eta^4). \tag{32}$$

Thus, this fourth order Leapfrog method has a phase gain instead of the usual phase lag. With this analysis which includes the time dependent terms the phase error behaves as a second-order method in contrast to the analysis of Kreiss and Oliger which ignores the time dependence. When the time step decreases we see that the error in Eq. (32) decreases as a cubic in the time step in contrast to the other methods discussed until now for which the error decreases linearly as the time step decreases. This indicates that this fourth order Leapfrog method would be more competitive at a smaller time step. Experience has shown however, that if the time steps are chosen too small then the errors accumulate and the total error is no longer small. Oliger has indicated that the optimal time step is approximately $\lambda\rho(A, B) = 1/4$ which is below the allowable time step based on stability criteria.

10. Crowley [6] has also introduced a scheme which uses points beyond the nine-point lattice in order to improve the phase error. For this scheme

$$\begin{aligned} w^{n+1/2} &= (I - \tfrac{1}{4}\lambda A\delta_x\mu_x + (\tfrac{1}{2}\lambda A)^2\,\delta_x^2)(1 - \tfrac{1}{4}\sigma B\delta_y\mu_y + (\tfrac{1}{2}\sigma B)^2\,\delta_y^2)\, w^n, \\ w^{n+1} &= w^n - (\lambda A\delta_x\mu_x + \sigma B\delta_y\mu_y)\, w^{n+1/2}. \end{aligned} \tag{33}$$

As with the splitting schemes one must alternate the order of the factor to preserve the second-order accuracy. The amplification matrix for this scheme with $\xi = \eta$ is

$$\begin{aligned} G(\xi, \xi) = I &- \tfrac{1}{2}(\lambda A + \sigma B)^2 \sin^2\xi(1 - \tfrac{1}{2}\lambda\sigma AB(1 - \cos\xi)) \\ &- i(\lambda A + \sigma B)\sin\xi\{1 - \tfrac{1}{2}(1 - \cos\xi)[(\lambda A)^2 + (\sigma B)^2 \\ &+ \tfrac{1}{2}\lambda\sigma AB(1 + \cos\xi) - \tfrac{1}{2}\lambda^2\sigma^2A^2B^2(1 - \cos\xi)]\}, \end{aligned} \tag{34}$$

and no stability proof was offered for the scheme due to its complexity. Also, for the special case of $\xi = \eta$, the phase error is

$$E(\xi, \xi) = \tfrac{1}{6}((\lambda A + \sigma B)\,\xi^3 - (\lambda A + \sigma B)^3\,\xi^3) + \tfrac{1}{4}(\lambda A + \sigma B) \times ((\lambda A)^2 + (\sigma B)^2 + \lambda\sigma AB)\,\xi^3 \tag{35}$$

The first step of this method is similar to the time splitting methods and this is reflected in the similarity of Eq. (35) to Eq. (20). Thus, this method of Crowley which extends beyond the nine-point lattice seems to offer no advantage over the nine-point time splitting method.

11. Historically the first two step two dimensional scheme was introduced by Richtmyer and has as its domain of dependence a nine-point diamond-shaped lattice [23]. Because of this nonrectangular shape the scheme has difficulties near boundaries. In addition there is a weak instability due to the lack of coupling between neighboring points (Kasahara [14]). Gourlay and Morris [10] have generalized this scheme by considering a family of methods that utilizes the thirteen points located within a diamond-shaped lattice with a diagonal length of two mesh widths. This scheme is given by

$$\begin{aligned} \bar{w} &= (\mu_x^2 + \mu_y^2 - 1)\,w^n - \alpha(\lambda\delta_x\mu_x f^n + \sigma\delta_y\mu_y g^n), \\ w^{n+1} &= w^n - \frac{1}{2\alpha}(\lambda\delta_x\mu_x\bar{f} + \sigma\delta_y\mu_y\bar{g}) - \left(1 - \frac{1}{2\alpha}\right)(\lambda\delta_x\mu_x f^n + \sigma\delta_y\mu_y g^n). \end{aligned} \tag{36}$$

We see that this method is similar to the generalized Burstein method [15] previously discussed although the domain of dependence of the schemes differ. The amplification matrix for this scheme is

$$G(\xi, \eta) = I - i(\lambda A \sin\xi + \sigma B \sin\eta)\left(1 - \frac{1}{2\alpha} + \frac{1}{4\alpha}(\cos\xi + \cos\eta)\right) - \frac{1}{2}(\lambda A \sin\xi + \sigma B \sin\eta)^2 \tag{37}$$

In this case the scheme is stable for all α with the stability condition given by

$$\rho^2(\lambda A, \sigma B) \leqslant \min_{\xi,\eta}\left\{\frac{4 - (1/4\alpha^2)[4\alpha - 2 + (\cos\xi + \cos\eta)^2]}{\sin^2\xi + \sin^2\eta}\right\}.$$

The phase error associated with the scheme is

$$E(\xi, \eta) = \frac{1}{6}(\lambda A\xi^3 + \sigma B\eta^3) - \frac{1}{6}(\lambda A\xi + \sigma B\eta)^3 + \frac{1}{8\alpha}(\lambda A\xi + \sigma B\eta)(\xi^2 + \eta^2) + O(\xi^4 + \eta^4). \tag{38}$$

As before we find that as α increases the phase lag decreases but the allowable time step also decreases. In any case this phase lag is larger than the corresponding one for the generalized Burstein method, Eq. (17). Thus, there does not seem to be any gain in using the larger lattice. In particular there seems to be no advantage of the original Richtmyer method over the rotated Richtmyer method (see also Wilson [27]).

IV. Results

The characterization of the numerical phase has thus far been limited to third order terms, i.e. terms of fourth order or higher in ξ and η have been neglected. We now consider the true numerical phase, $\arctan((\text{Im } G)(\text{Re } G)^{-1})$. In order to compare the phase representations of the schemes we restrict ourselves to the scalar case with $A = B$ and choose the time step so that $\lambda A = 1/4$ which is within the stability range of most of the schemes considered. In order for the plots to be two-dimensional (a comparison of many three-dimensional perspective plots would be difficult to decipher) we have only considered the case $\xi = \eta$. With these restrictions the amplification matrix of the MacCormack scheme (Eq. (15)) reduces to that of the Lax–Wendroff scheme (Eq. (10)) and so it will not be treated separately. In Fig. 1 we compare the analytic phase together with the true numerical phase for the nine-point schemes considered above. The schemes considered in Fig. 1 are as follows: (1) analytic phase error, (2) first order Lax scheme (see Ref. [16]), (3) Strang splitting scheme $L_x L_y$ [Eq. (19)], (4) Lax–Wendroff [Eq. (9)], (5) Leapfrog [Eq. (22)], and the generalized Burstein scheme [Eq. (15)] with (6) $\alpha = 1/4$, (7) $\alpha = 1/2$, (8) $\alpha = 1$, (9) $\alpha = 100$. All these schemes are considered without any artificial fourth order viscosities, i.e. $\nu = 0$. These graphs confirm our previous assertion that the Leapfrog and Lax–Wendroff methods have similar phase errors. The phase lag for the generalized Burstein method is not as small but improves as α increases. With $\alpha = 100$ the phase error is virtually indistinguishable from that of the Leapfrog scheme. As is to be expected the first order Lax method is clearly inferior to the second order methods. In fact the phase error is infinite at $\xi = \pi/2$, and is shown in the plot as finite due to a cutoff to prevent offscale plotting. In Table I is given the true numerical phase for these schemes, at $\xi = \pi/5$.

The introduction of a fourth order viscosity term in Eq. (15) clearly has had no effect on our phase analysis which considers only terms through third order. Nevertheless, Figs. 2 and 3 demonstrate that this viscosity can have a profound effect on the phase error for large ξ. In Fig. 2 is plotted the true numerical phase for (1) analytic solution, (2) Lax–Wendroff, and the generalized Burstein scheme

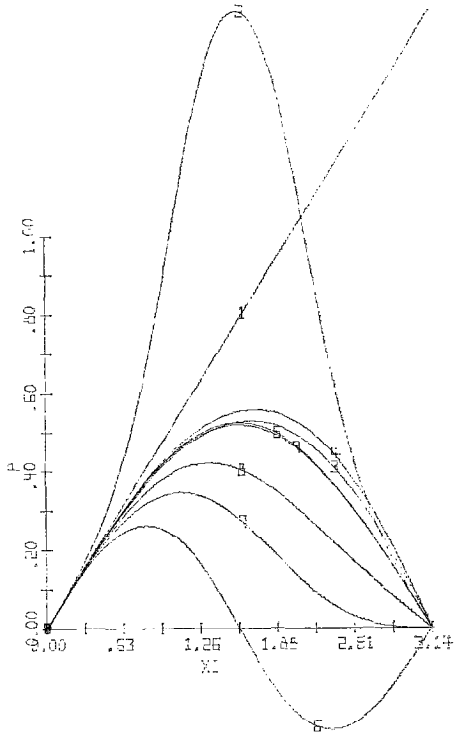


FIG. 1. Complete numerical phase of various nine-point schemes. See text for details.

TABLE I

Scheme	Phase at $\xi = \pi/5$
1. Analytic	0.31416
2. Lax	0.42205
3. Splitting	0.29528
4. Lax-Wendroff	0.29868
5. Leapfrog	0.29830
6. Burstein $\alpha = 1/4$	0.24356
7. " $\alpha = 1/2$	0.27099
8. " $\alpha = 1$	0.28455
9. " $\alpha = 100$	0.29788

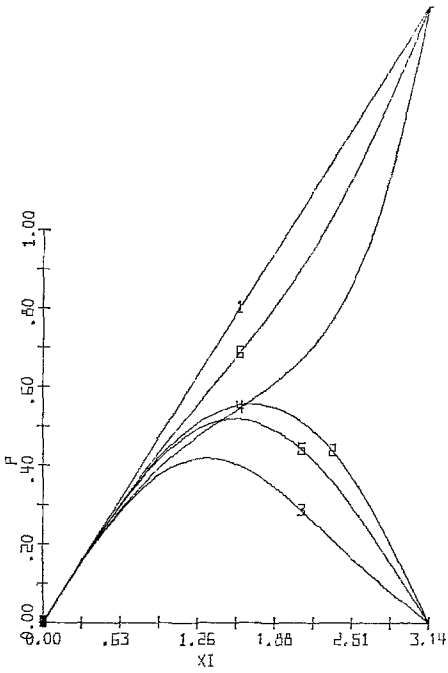


FIG. 2. Complete numerical phase for generalized Burstein scheme with a viscosity coefficient of 0.0625. See text for details.

with (3) $\alpha = 1$, $\nu = 0$, (4) $\alpha = 1$, $\nu = 0.0625$, (5) $\alpha = 100$, $\nu = 0$, (6) $\alpha = 100$, $\nu = 0.0625$. Figure 3 is similar to Fig. 2 with $\nu = 0.125$ in plots 4 and 6. We note that only plots 4 and 6 are new in these graphs with the others plotted for comparison's sake only. In Fig. 2 we see that with $\nu = 0.0625$ there is zero phase error at $\xi = \pi$ while all previous cases showed no numerical motion at all to these short waves. However, for small ξ the effect of the viscosity term is small as was expected. As seen in the numerical experiments (to be described) the effect of the viscosity terms on the phase error is rather small which confirms our assumption that the phase error is meaningful only for ξ small and that even complete agreement for the high frequency waves does not improve the computational phase error. For completeness Figs. 4, 5 and 6 show the magnitude of the amplification factors corresponding to Figs. 1, 2 and 3 respectively.

As a computational test of the nine-point schemes previously discussed we have chosen a sample problem from linear elasticity. This problem has the advantage of having constant coefficients so that Fourier analysis is applicable.

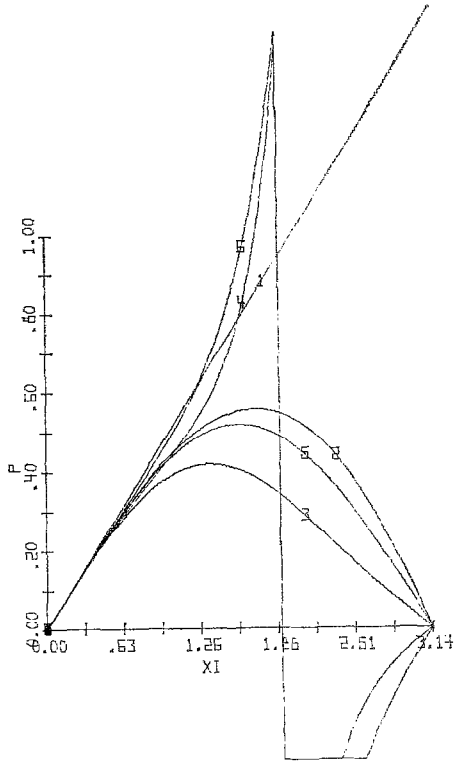


FIG. 3. Similar to Fig. 2 but with a viscosity coefficient of 0.125 for those schemes containing an artificial viscosity.

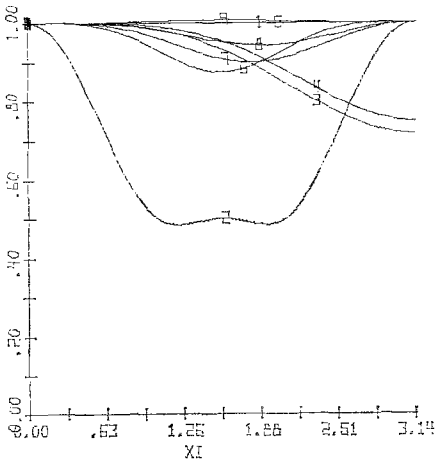


FIG. 4. Norm of the amplification matrix for those schemes appearing in Fig. 1.

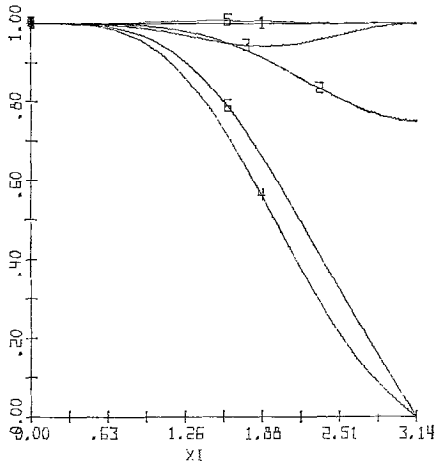


FIG. 5. Norm of the amplification matrix for those schemes appearing in Fig. 2.

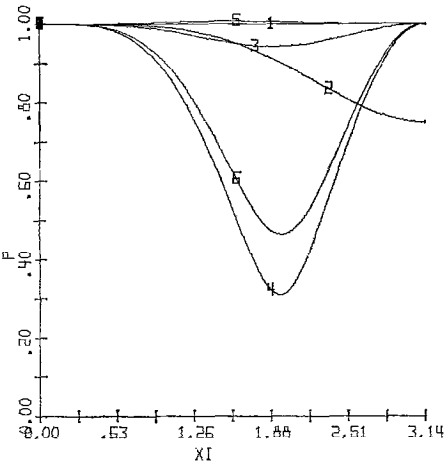


FIG. 6. Norm of the amplification matrix for those schemes appearing in Fig. 3.

Furthermore, because of the simplicity of the equations, analytic solutions are known so that a meaningful comparison of schemes can be made. Yet the problem is realistic in that a system of five coupled equations is given rather than a scalar equation, and the matrices A and B do not commute. Also wave propagation occurs only in the x direction so that the x and y directions are not treated symmetrically.

The equations are those of linear elasticity, i.e.,

$$\begin{aligned} \rho u_t &= \tau_{11,x} + \tau_{12,y}, \\ \rho v_t &= \tau_{12,x} + \tau_{22,y}, \\ \tau_{11,t} &= (2\mu + \lambda) u_x + \lambda v_y, \\ \tau_{12,t} &= \mu(u_y + v_x), \\ \tau_{22,t} &= \lambda u_x + (2\mu + \lambda) v_y, \end{aligned} \tag{39}$$

where ρ, λ, μ are positive constants. These equations are integrated in the domain $0 \leqslant x \leqslant \Omega$, and $0 \leqslant y \leqslant b$ with periodicity conditions in the x direction and symmetry for u, τ_{11}, τ_{12} about the axis $y = 0$ and antisymmetry for v, τ_{12}. Thus we are considering the propagation of a symmetric wave which is travelling down an infinite bar. At $y = b$ the boundary condition is the free surface requirement that the normal and shear stresses be zero, i.e., $\tau_{12} = \tau_{22} = 0$. The sound speeds associated with (39) are $\pm((\lambda + 2\mu)/\rho)^{1/2}$, $\pm(\mu/\rho)^{1/2}$, 0.

A particular solution to this problem is

$$\begin{aligned} u(x, y, t) &= \omega B \left(\xi \cos \alpha y + \frac{(\beta^2 - \xi^2) \cos \alpha b}{2\xi \cos \beta b} \cos \beta y\right) \cos \xi(x - vt), \\ v(x, y, t) &= -\omega B \left(\alpha \sin \alpha y - \frac{(\beta^2 - \xi^2) \cos \alpha b}{2\beta \cos \beta b} \sin \beta y\right) \sin \xi(x - vt), \\ \tau_{11}(x, y, t) &= \mu B \left((2\alpha^2 - \beta^2 - \xi^2) \cos \alpha y - \frac{(\beta^2 - \xi^2) \cos \alpha b}{\cos \beta b} \cos \beta y\right) \\ &\quad \times \cos \xi(x - vt), \\ \tau_{12}(x, y, t) &= 2\mu\alpha\xi B \left(\sin \alpha y - \frac{\sin \alpha b}{\sin \beta b} \sin \beta y\right) \sin \xi(x - vt), \\ \tau_{22}(x, y, t) &= \mu(\xi^2 - \beta^2) B \left(\cos \alpha y - \frac{\cos \alpha b}{\cos \beta b} \cos \beta y\right) \cos \xi(x - vt), \end{aligned} \tag{40}$$

where the parameters appearing in Eq. (40) have the following meaning

$$\begin{aligned} \xi &= \frac{2\pi}{\Omega}, \qquad \omega = \xi v, \\ \alpha^2 &= \frac{\omega^2}{c_1^2} - \xi^2 = \xi^2 \left(\frac{v^2}{c_1^2} - 1\right), \qquad \beta^2 = \frac{\omega^2}{c_2^2} - \xi^2 = \xi^2 \left(\frac{v^2}{c_2^2} - 1\right). \end{aligned} \tag{41}$$

The sound speeds c_1, c_2 are given by

$$c_1^2 = (\lambda + 2\mu)/\rho, \qquad c_2^2 = \mu/\rho.$$

B is an arbitrary constant and α and β are complex numbers. The free surface boundary conditions are satisfied if v is a solution to the transcendental equation

$$\frac{\tan \beta b}{\tan \alpha b} + \frac{4\xi^2 \alpha\beta}{(\xi^2 - \beta^2)^2} = 0, \tag{42}$$

with $\alpha = \alpha(v)$ and $\beta = \beta(v)$ given by Eq. (41). The reader is referred to Mindlin [21] for further details about this and similar problems.

If one chooses as initial conditions Eq. (40) with $t = 0$ then Eq. (40) is the unique solution to the given initial boundary valued problem. The parameters used for the test problem were

$$B = 1, \qquad \Omega = b = 1, \qquad \rho = 0.175, \qquad \lambda = 0.3, \qquad \mu = 0.2.$$

With $\xi = 2\pi/\Omega$ the wave length of the solution is equal to one period. Δx and Δy were both chosen as 1/16. With these parameters the sound speeds are approximately

$$c_1 = 2.0, \qquad c_2 = 1.069.$$

The third mode was chosen from Eq. (40) which yields a value for v of approximately 1.6975 which implies that α is an imaginary number with β real. A short bar, $\Omega = b$, together with a low mode solution to Eq. (42) were chosen. This combination of parameters increases wave dispersion in the solution compared to the wave motion in the longer bar which was considered by Clifton [5]. Since long waves are less subject to dispersion the problem described in [5] is not as severe a test of the numerical phase error accumulation generated by a difference scheme.

Comparing the various nine point schemes after four complete periods we consider the following characteristics of the solutions in Tables II through V. Let

$$\text{CFL} = \frac{\Delta t}{\Delta x}\rho(A, B) = \frac{\Delta t}{\Delta x}\left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}.$$

The time step Δt is constant throughout the time integration and is chosen so that a complete period takes an integer number of time steps. Let

$$\text{WCHG} = \frac{\text{initial energy} - \text{energy at time } t}{\text{initial energy}},$$

where the elastic energy of the system is given by

$$W = \frac{\rho}{2}(u^2 + v^2) + \frac{1}{2\mu(3\lambda + 2\mu)}[(\lambda + \mu)(\tau_{11}^2 + \tau_{22}^2) - \lambda\tau_{11}\tau_{22} + (3\lambda + 2\mu)\tau_{12}^2].$$

WCHG measures the L_2 stability of the method; when WCHG is negative the norm of the numerical solution is increasing in time in contrast to the norm of

the analytic solution which is independent of time. When the energy has doubled within the four periods of the solution (160 time steps at CFL = 0.471) then the method is classified as unstable for the time step used. Let

$$\text{PHASE} = \frac{\text{pos(analytic)} - \text{pos(numeric)}}{\text{pos(analytic)}}$$

where pos denotes the position of the zero of the u component of velocity. The numerical position of the zero is calculated by linear interpolation for a particular y value and then averaged over all the y coordinate lines. A fourth order viscosity term of the form $-\nu\delta_x^2\delta_y^2 w$, $0 \leqslant \nu \leqslant 1/8$, was also included for the schemes considered.

The schemes considered in Tables II–V are:

LW—Lax–Wendroff method, Eq. (9),
LF—Leapfrog method, Eq. (22),
S—Strang splitting method $L_xL_yL_yL_x$, Eqs. (18) and (19),
B1—Generalized Burstein scheme with $\alpha = \frac{1}{4}$, Eq. (15),
B2—Generalized Burstein scheme with $\alpha = \frac{1}{2}$ (rotated Richtmyer),
B3—Generalized Burstein scheme with $\alpha = 1$ (original Burstein),
B4—Generalized Burstein scheme with $\alpha = 100$,
M1—MacCormack method using forward differences in first step, Eq. (12),
M2—MacCormack method using forward x and backward y differences.
M3—MacCormack method using backward x and forward y differences.
M4—MacCormack method using backward differences in first step,
M5—rotational use of M1, M2, M3, M4 at successive time steps.

We see from Table II that even with relatively small time steps the time splitting scheme has a phase error as small as any of the other schemes. The Lax–Wendroff method remains stable even though we have exceeded CFL = $1/(8)^{1/2}$. Note that the Leapfrog scheme has a phase error very close to that of the Lax–Wendroff method. The phase error for the rotated Richtmyer method is quite large while that for the Burstein scheme ($\alpha = 1$) is intermediate. The MacCormack method displays quite different properties depending on which variant is chosen. Even the alternation of the variants yields a scheme which is beginning to go unstable, though it can be stabilized by the addition of viscosity terms. In this case the phase error is comparable to that of the Burstein scheme.

Table III is also based on data from the elastic wave propagation problem but with a larger time step. The Lax–Wendroff and Leapfrog methods have now both become unstable while the splitting scheme remains stable and even has a smaller phase lag than in Table II. The generalized Burstein scheme is stable if α is chosen sufficiently small but the phase error is very large, while the MacCormack scheme has also become unstable. Thus, although the MacCormack scheme involves few

TABLE II

CFL = 0.471

Scheme	ν	WCHG	Phase
LW	0.	0.1299	0.027
LF	0.	−0.0088	0.029
S	0.	0.1755	0.026
B1	0.	0.3432	0.109
B2	0.	0.1714	0.068
B3	0.	0.0345	0.047
B4	0.	−0.1209	0.028
B4	0.0625	0.3960	0.026
M1	0.	0.1527	0.026
M2	0.	unstable	
M2	0.0625	unstable	
M3	0.	unstable	
M3	0.0625	unstable	
M4	0.	0.7846	−0.143
M5	0.	−0.5501	0.045
M5	0.0625	0.0937	0.044

TABLE III

CFL = 0.754

Scheme	ν	WCHG	Phase
LW	0.	−0.7484	0.020
LW	0.0625	unstable	
LF	0.	unstable	
S	0.	0.1785	0.020
B1	0.	0.4704	0.101
B2	0.	0.2086	0.060
B3	0.	−0.0278	0.042
B3	0.0625	0.3003	0.040
B4	0.	unstable	
B4	0.0625	−0.8164	0.021
M1	0.	−0.3637	0.023
M1	0.0625	unstable	
M4	0.	−0.8457	−0.155
M5	0.	unstable	
M5	0.0625	unstable	

function evaluations (though not less than L_xL_y) it suffers from the need of a small time step. We also see from this example that the added viscosity term can, at times, cause a decrease in the stability of the solution.

TABLE IV

CFL = 0.897

Scheme	ν	WCHG	Phase
S	0.	0.0044	0.019
B1	0.	0.4950	0.095
B2	0.	0.0629	0.060
B2	0.0625	0.3577	0.057
B3	0.	unstable	
B3	0.0625	unstable	

With a time step of almost CFL = 0.9 we see in Table IV that only the time splitting scheme and the generalized Burstein scheme $\alpha = 1/4, 1/2$ remain stable without the use of viscosity terms, though as before the phase error for the case $\alpha = 1/4$ remains large. When we increase the time step to CFL = 0.947 even the Strang splitting scheme becomes unstable as seen in Table V. This is in agreement

TABLE V

CFL = 0.942

Scheme	ν	WCHG	Phase
S	0.	−0.3400	0.021
B1	0.	0.4945	0.094
B2	0.	−0.5203	0.052
B2	0.0625	0.0835	0.059
B3	0.	unstable	

with the experiments of Gourlay and Morris [11] who also found instabilities in another splitting method for CFL greater than 0.9; this may possibly be due to the boundary treatment. Thus for the dynamic elastic equations the generalized Burstein scheme with $\alpha = 1/4$ allows the largest computational time step (though we have shown that it is unconditionally unstable for other systems of equations). The results of these tables are confirmed by many similar computer runs that were completed but not included due to lack of space.

These results were calculated using space extrapolation, for the undefined variables, at the boundary $y = b$, which is the only boundary where extra condi-

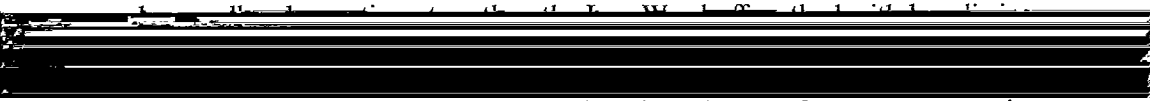
tions need be imposed. Characteristic methods (see Clifton [5]) were also tried, for several schemes, but with little effect on the phase errors. It thus seems that correct boundary treatment is more crucial for stability than for phase representations.

V. Conclusions

When one begins to solve a new problem one always encounters the question of which difference scheme to use. With this study we assume that the choice has been limited to explicit Eulerian schemes. With the high-speed computers presently available there seems to be no advantage to first order codes and hence only higher order methods need to considered.

For smooth problems where dissipation is not wanted the Leapfrog method seems to be an ideal method. Where boundaries do not cause any complications, e.g. global circulation models, the staggered Leapfrog or the fourth order (in space) Leapfrog of Kreiss and Oliger can be used, otherwise the nonstaggered Leapfrog offers a good phase representation with a reasonable time step. However, even with smooth problems one must be careful of nonlinear instabilities (see Fornberg [8]). For problems with shocks an artificial viscosity is needed and it seems more natural to go to dissipative schemes where the viscosity is built in and there is no need to guess at the appropriate values of the constants which may appear in the artificial viscosity.

Among the nine-point schemes considered the time splitting scheme [Eqs. (18) and (19)] offers the maximal time step together with a phase lag as small as any of the other schemes considered. Thus, both from the viewpoints of computer speed and decreased phase errors this scheme seems the most appropriate; i.e., it works for problems where velocities are important and for asymptotic type problems where long computing times exist but phase errors are relatively unimportant. For cases where time-dependent boundary conditions may cause difficulties for the splitting technique the Lax–Wendroff method, written as a two-step method [Eq. (9)], has a small phase error but also requires a small time step. The methods of Burstein and MacCormack can display instabilities for particular choices of the matrices A and B, and so care must be exercised when using these schemes without additional artificial viscosity. In many cases however these



Lax–Wendroff or splitting schemes and should not be used for wave propagation problems.

With periodic boundaries or simple boundary conditions it may be advisable to use schemes that require data beyond the nine-point lattice. The most obvious

approach is to use methods that are third or fourth order in space and time or at least fourth order in space only. Among the second-order schemes that go beyond the nine-point rectangular lattice are the family of schemes introduced by Gourlay and Morris, Eq. (34). This family, which includes the original Richtmeyer method, has a larger phase error than most of the nine-point schemes and hence should never be used for wave propagation problems. The scheme introduced by Boris and Book requires a large amount of computation per time step and is not really second order due to the introduction of nonlinear cutoffs. Thus, it is at a distinct disadvantage when compared with third- or fourth-order schemes. Nevertheless, for problems where compression of the shock region is important this method should prove useful. The scheme introduced by Fromm is also a nonlinear scheme even for linear problems and it also suffers from the requirement of small permissible time steps. Hence, here too it is not clear that there is any computational advantage over higher order schemes which do not have a larger domain of dependence and give dramatically better phase representations. There also seems to be no advantage to the splitting technique $L_{x/2}L_yL_{x/2}$ which requires more data and more function evaluations than the splitting L_xL_y described in Eq. (19).

The conclusions reached above strictly hold only for linear problems. In the nonlinear case the phase error is no longer independent of the dissipation introduced by the scheme and so is problem dependent. Thus, both the Burstein and MacCormack schemes have provided answers to many fluid dynamic problems even though both may suffer from weak instabilities. Both the two-step Lax–Wendroff (Thommen) method and the Burstein scheme were applied to a complicated problem in fluid dynamics with moving boundaries. The velocities predicted by the Lax–Wendroff method differed from those of the Burstein method by only about one percent using a coarse mesh of about 10×15 in the original domain, with the Lax–Wendroff method being closer to the experimental value. Thus, in this particular case the difference between the two schemes was negligible, although it was not for the linear elastic test problem. Nevertheless the results for the linear problems should at least provide a guide for the selection of a scheme for nonlinear problems. At the very least it is impractical to try each scheme for a particular nonlinear problem to discover which scheme is best for that particular problem. Thus, at least on practical grounds the conclusions reached can be applied to nonlinear as well as linear problems. Thus, the nine point Strang splitting method is a most reasonable first try in solving any problem followed by Leapfrog [Eq. (22) or (28)] or the Thommen scheme [Eq. (9)]. If a small phase error is of great importance then one should use higher order schemes, which of necessity use data beyond a nine-point rectangular lattice. It also appears that the larger a time step one uses, within the scheme's stability range, the smaller the phase error. This is a phenomenon which is well known for shocks but is also true for smooth problems.

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